Automorphism Groups of Centralizers of Idempotents

João Araújo
Universidade Aberta, R. Escola Politécnica, 147
1269-001 Lisboa, Portugal
&
Centro de Álgebra, Universidade de Lisboa,
1649-003 Lisboa, Portugal, mjoao@lmc.fc.ul.pt

Janusz Konieczny
Department of Mathematics, Mary Washington College,
Fredericksburg, VA 22401, USA, jkoniecz@mwc.edu

Abstract

For a set X, an equivalence relation ρ on X, and a cross-section R of the partition X/ρ, consider the following subsemigroup of the semigroup $T(X)$ of full transformations on X:

$$T(X,\rho,R)=\{a\in T(X):Ra\subseteq R \text{ and } (x,y)\in\rho\Rightarrow(xa,ya)\in\rho\}.$$

The semigroup $T(X,\rho,R)$ is the centralizer of the idempotent transformation with kernel ρ and image R. We prove that the automorphisms of $T(X,\rho,R)$ are the inner automorphisms induced by the units of $T(X,\rho,R)$ and that the automorphism group of $T(X,\rho,R)$ is isomorphic to the group of units of $T(X,\rho,R)$.

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1 Introduction

Let X be an arbitrary nonempty set. The semigroup $T(X)$ of full transformations on X consists of the functions from X to X with composition as the semigroup operation. It has the symmetric group $S(X)$ of permutations on X as its group of units and it is a subsemigroup of the semigroup $PT(X)$ of partial transformations on X.

Let S be a transformation semigroup on X (a subsemigroup of $PT(X)$). The group of automorphisms of S will be denoted by $Aut(S)$. A subset T of S is called *characteristic in* S if $\phi(T)=T$ for every $\phi\in Aut(S)$. An automorphism ϕ of S is called *inner* if there is $g\in S(X)$ such that $\phi(a)=g^{-1}ag$ for every $a\in S$. (In such a case, we shall say that ϕ is the *inner automorphism induced by* g.)

Automorphisms of transformation semigroups have been studied by various authors. Schreier [7] and Mal'cev [6] proved that the automorphisms of $T(X)$ are the inner automorphisms induced by the elements of $S(X)$ and that $Aut(T(X))$ is isomorphic to $S(X)$. Similar results have been obtained for the semigroup $PT(X)$ by Šutov [9] and Magill [5], and for the symmetric inverse semigroup $I(X)$ of all partial one-to-one transformations on X by Liber [4]. Sullivan [8] and Levi [2], [3] generalized the above results to the class of $S(X)$-normal semigroups (that is, semigroups closed under conjugations by permutations).

Let ρ be an equivalence relation on X and let R be a cross-section of the partition X/ρ induced by ρ. Consider the semigroup

$$T(X,\rho,R) = \{a \in T(X) : Ra \subseteq R \text{ and } (x,y) \in \rho \Rightarrow (xa,ya) \in \rho\}.$$

Our aim is to determine the automorphism group of $T(X,\rho,R)$ (for arbitrary ρ and R). If $\rho = \Delta = \{(x,x) : x \in X\}$ and $R = X$ then $T(X,\Delta,X) = T(X)$. If $\rho = X \times X$ and $R = \{r\}$ then $T(X,\rho,R)$ is isomorphic to $PT(X')$ where $X' = X - \{r\}$. Thus our results generalize the results of Schreier [7] and Mal'cev [6] on the automorphism group of $T(X)$, and of Šutov [9] and Magill [5] on the automorphism group of $PT(X)$. We point out that in general $T(X,\rho,R)$ is not an $S(X)$-normal semigroup.

The organization of the paper is as follows. In Section 2, we prove that the semigroups $T(X,\rho,R)$ are the same as the centralizers of idempotent transformations. In Section 3, we describe the group of units of $T(X,\rho,R)$. In Section 4, we prove that the sets T_1 (of all elements of $T(X,\rho,R)$ of rank 1), T_2 (of all elements of $T(X,\rho,R)$ of rank 2), and two $\mathcal{J}$-classes into which T_2 is partitioned are characteristic in $T(X,\rho,R)$. In Section 5, we introduce right inverse diagonal bands and determine which of them are maximal. Finally, in Section 6, we use the results of the previous sections to prove the main theorems of the paper: the automorphisms of $T(X,\rho,R)$ are the inner automorphisms induced by the units of $T(X,\rho,R)$ (Theorem 6.4) and $Aut(T(X,\rho,R))$ is isomorphic to the group of units of $T(X,\rho,R)$ (Theorem 6.6).

2 Semigroups $T(X,\rho,R)$ and Centralizers of Idempotents

For $a \in T(X)$, we denote the kernel of a (the equivalence relation $\{(x,y) \in X \times X : xa = ya\}$) by $\mathrm{Ker}(a)$ and the image of a by ∇a. The cardinality of ∇a is called the *rank* of a. For $Y \subseteq X$, Ya will denote the image of Y under a, that is, $Ya = \{xa : x \in Y\}$. Note that we write transformations (which map elements of X to elements of X) on the right (that is, xa instead of $a(x)$). On the other hand, we write automorphisms (which map transformations to transformations) on the left (that is, $\phi(a)$). Moreover, transformations will always be represented by Roman letters and automorphisms by Greek letters. Let ρ be an equivalence relation on X and R a cross-section of X/ρ. If $x \in X$ then there is exactly one $r \in R$ such that $x\,\rho\,r$, which will be denoted by r_x. Of course, for $s \in R$, we have $r_s = s$.

Lemma 2.1 *Let $a \in T(X,\rho,R)$ and $x \in X$. Then $r_x a = r_{xa}$.*

Proof: By the definition of r_x, $(x,r_x) \in \rho$ and hence $(xa,r_x a) \in \rho$, with $r_x a \in R$. Moreover, $(xa,r_{xa}) \in \rho$ and $r_{xa} \in R$. Now $(r_x a,xa),(xa,r_{xa}) \in \rho$ implies $(r_x a,r_{xa}) \in \rho$ and hence $r_x a = r_{xa}$ (because $r_x a, r_{xa} \in R$). ■

Let S be a semigroup and $a \in S$. The *centralizer* of a is defined as

$$C(a) = \{b \in S : ab = ba\}.$$

It is clear that $C(a)$ is a subsemigroup of S.

The full transformation semigroup $T(X)$ is the centralizer of id_X, the identity mapping on X, that is, $C(id_X) = T(X) = T(X, \Delta, X)$. We generalize this observation by showing that the semigroups $T(X, \rho, R)$ are precisely the centralizers of idempotent transformations.

Let $e \in T(X)$ be an idempotent ($e^2 = e$). Then for every $x \in X$, $(xe)e = x(ee) = xe$. It follows that $R = \nabla e$ consists of the fixed points of e and it is a cross-section of the partition X/ρ induced by the equivalence relation $\rho = \operatorname{Ker}(e)$.

Lemma 2.2 *Let $e \in T(X)$ be an idempotent with $\rho = \operatorname{Ker}(e)$ and $R = \nabla e$. Then $T(X, \rho, R) = C(e)$.*

Proof: Suppose $a \in T(X, \rho, R)$ and $x \in X$. Observe that, by the definition of r_x, $xe = r_x e = r_x$. Therefore, by Lemma 2.1,

$$x(ae) = (xa)e = r_{xa} = r_x a = (xe)a = x(ea).$$

It follows that $ae = ea$ and so $a \in C(e)$.

Conversely, suppose $a \in C(e)$, that is, $ea = ae$. Then for all $x, y, r \in X$:

$(x, y) \in \rho \Rightarrow xe = ye \Rightarrow (xe)a = (ye)a \Rightarrow (xa)e = (ya)e \Rightarrow (xa, ya) \in \rho = Ker(e)$ and

$r \in R \Rightarrow re = r \Rightarrow (re)a = ra \Rightarrow (ra)e = ra \Rightarrow ra \in \nabla e = R.$

Thus $a \in T(X, \rho, R)$. ■

Theorem 2.3 *Let S be a subsemigroup of $T(X)$. Then the following are equivalent:*

(1) *$S = T(X, \rho, R)$ for some equivalence relation ρ on X and some cross-section R of X/ρ.*

(2) *There exists an idempotent $e \in T(X)$ such that $S = C(e)$.*

Proof: Suppose $S = T(X, \rho, R)$ where ρ and R are as in (1). Define $e \in T(X)$ by $xe = r_x$. Then e is an idempotent with $\operatorname{Ker}(e) = \rho$ and $\nabla e = R$, and so $S = C(e)$ by Lemma 2.2.

Conversely, if $S = C(e)$ for some idempotent $e \in T(X)$ then, by Lemma 2.2, $S = T(X, \rho, R)$ where $\rho = \operatorname{Ker}(e)$ and $R = \nabla e$. ■

For the remainder of the paper, ρ will denote an equivalence relation on X and R will denote a cross-section of X/ρ.

3 Group of Units of $T(X, \rho, R)$

In this section we describe the group of units of $T(X, \rho, R)$. Consider the following subset of $T(X, \rho, R)$:

$$S(X, \rho, R) = \{g \in S(X) : Rg = R \text{ and } (x, y) \in \rho \Leftrightarrow (xg, yg) \in \rho\}.$$

It is clear that $id_X \in S(X, \rho, R)$, $S(X, \rho, R)$ is closed under composition, and $S(X, \rho, R)$ is closed under inverses (that is, if $g \in S(X, \rho, R)$ then $g^{-1} \in S(X, \rho, R)$). It follows that $S(X, \rho, R)$ is a subgroup of $T(X, \rho, R) \cap S(X)$.

Theorem 3.1 *Let $g \in T(X, \rho, R)$. Then the following are equivalent:*

(1) *g is a unit of $T(X, \rho, R)$.*

(2) *$g \in T(X, \rho, R) \cap S(X)$.*

(3) *$g \in S(X, \rho, R)$.*

Proof: (1) $\Rightarrow$ (2). Suppose g is a unit of $T(X, \rho, R)$. Then there is $g' \in T(X, \rho, R)$ such that $gg' = g'g = id_X$. Thus $g \in S(X)$ and so $g \in T(X, \rho, R) \cap S(X)$.

(2) $\Rightarrow$ (3). Suppose $g \in T(X, \rho, R) \cap S(X)$. Since $g \in T(X, \rho, R)$, $Rg \subseteq R$. To prove that $R \subseteq Rg$, let $p \in R$. Since g is onto, $xg = p \in R$ for some $x \in X$. As $r_x \in R$, it follows that $r_x g \in R$. Moreover, $(x, r_x) \in \rho$ implies $(xg, r_x g) \in \rho$. Thus $xg = r_x g$ (because they are ρ-related and they both belong to R) and hence $p = xg = r_x g \in Rg$. It follows that $Rg = R$.

Let $x, y \in X$. Since $g \in T(X, \rho, R)$, $(x, y) \in \rho \Rightarrow (xg, yg) \in \rho$. For the reverse implication, suppose $(xg, yg) \in \rho$. Then $r_{xg} = r_{yg}$ and hence, by Lemma 2.1,

$$r_x g = r_{xg} = r_{yg} = r_y g.$$

It follows that $r_x = r_y$ (because g is one-to-one) and so $(x, y) \in \rho$.

(3) $\Rightarrow$ (1). Let $g \in S(X, \rho, R)$. Then $g^{-1} \in S(X, \rho, R)$ and so $g, g^{-1} \in T(X, \rho, R)$ and $gg^{-1} = g^{-1}g = id_X$. Thus g is a unit of $T(X, \rho, R)$. ■

4 Some Characteristic Subsets of $T(X, \rho, R)$

Let S be a semigroup. Recall that $T \subseteq S$ is characteristic in S if $\phi(T) = T$ for every $\phi \in Aut(S)$.

If S is a monoid and $a, b \in S$, we say that $a \mathscr{J} b$ if $a = cbd$ and $b = c'ad'$ for some $c, d, c', d' \in S$. The relation $\mathscr{J}$ is one of the five equivalences on S known as *Green's relations* [1, p. 45].

For an integer $n \geq 1$, define

$$T_n = \{a \in T(X, \rho, R) : |\nabla a| = n\}.$$

Note that T_1 is a subsemigroup of $T(X, \rho, R)$.

We shall need the fact that T_1 is characteristic in every subsemigroup S of $T(X, \rho, R)$ in which it is included. For semigroups T and S, we write $T \leq S$ if T is a subsemigroup of S.

Lemma 4.1 *Let S be a semigroup such that $T_1 \leq S \leq T(X, \rho, R)$ and let $a \in S$. Then $a \in T_1$ if and only if $a = ba$ for every $b \in S$.*

Proof: Suppose $a \in T_1$ and $b \in S$. Then $\nabla a = \{r\}$ for some $r \in R$. Thus for every $x \in X$, $x(ba) = (xb)a = r = xa$, and so $ba = a$.

Conversely, suppose $a = ba$ for every $b \in S$. Fix $r \in R$ and consider $b \in T(X, \rho, R)$ such that $xb = r$ for every $x \in X$. Note that $b \in T_1 \leq S$. Thus $a = ba$ and so $\nabla a = \nabla(ba) = \{ra\}$, implying $a \in T_1$. ■

Theorem 4.2 *Let S be a semigroup such that $T_1 \leq S \leq T(X, \rho, R)$. Then T_1 is characteristic in S.*

Proof: Let $\phi \in Aut(S)$. Then, by Lemma 4.1, for every $a \in S$:

$$\begin{aligned} a \in T_1 &\Leftrightarrow a = ba \text{ for every } b \in S \\ &\Leftrightarrow \phi(a) = \phi(b)\phi(a) \text{ for every } b \in S \\ &\Leftrightarrow \phi(a) = c\phi(a) \text{ for every } c \in S \\ &\Leftrightarrow \phi(a) \in T_1. \end{aligned}$$

Thus $\phi(T_1) = T_1$ and so T_1 is characteristic in S. ■

In T_2 we have elements whose image is $\{r, x\}$ with $r \in R$, $x \in X - R$, and $(x, r) \in \rho$; and elements whose image is $\{r, s\}$ with $r, s \in R$. We denote the set of elements of the former kind by J_1, and the set of elements of the latter kind by J_2. It is clear that $T_2 = J_1 \cup J_2$ and $J_1 \cap J_2 = \emptyset$.

The following two lemmas prove that J_1 is either empty or it is a $\mathcal{J}$-class of $T(X, \rho, R)$, and that the same result is true for J_2.

Lemma 4.3 *Let $a \in J_1$ and $b \in T(X, \rho, R)$. Then $b \in J_1$ if and only if $a \mathcal{J} b$.*

Proof: Suppose $b \in J_1$. Then $\nabla a = \{r, x\}$ and $\nabla b = \{s, y\}$ for some $r, s \in R$ and $x, y \in X - R$. For $t \in R$, let $A(r, t) = \{z \in t\rho : za = r\}$ and $A(x, t) = \{z \in t\rho : za = x\}$. Since $b \in T(X, \rho, R)$, there are $s_0 \in R$ and $y_0 \in s_0\rho$ such that $s_0 b = s$ and $y_0 b = y$. We define $c, d \in T(X)$ as follows. For $t \in R$, define c so that $A(r, t)c = \{s_0\}$ and $A(x, t)c = \{y_0\}$. Define d so that $(X - \{y\})d = \{r\}$ and $yd = x$.

By the definition of $A(r, t)$ and $A(x, t)$ and the construction of c and d, we have $c, d \in T(X, \rho, R)$. Now let $z \in X$. Then $z \in t\rho$ for some $t \in R$. If $z \in A(r, t)$ then $za = r$ and $z(cbd) = s_0(bd) = sd = r$. If $z \in A(x, t)$ then $za = x$ and $z(cbd) = y_0(bd) = yd = x$. Thus $a = cbd$. By symmetry, $b = c'ad'$ for some $c', d' \in T(X, \rho, R)$, and so $a \mathcal{J} b$.

Conversely, suppose that $a \mathcal{J} b$. Then $a = cbd$ for some $c, d \in T(X, \rho, R)$, which implies that $|\nabla a| = |\nabla(cbd)| \leq |\nabla b|$. By symmetry, $|\nabla b| \leq |\nabla a|$, and so $|\nabla b| = |\nabla a| = 2$. Moreover, since $a \in J_1$, $\nabla a = \{x, r\}$ where $x \in X - R$, $r \in R$, and $(x, r) \in \rho$. Thus, since $b = c'ad'$ for some $c', d' \in T(X, \rho, R)$, $\nabla b = \nabla(c'ad') \subseteq \{xd', rd'\}$. Since $d' \in T(X, \rho, R)$, $(xd', rd') \in \rho$, which implies that $|\{xd', rd'\} \cap R| = 1$. It follows that $b \in J_1$. ■

Lemma 4.4 *Let $a \in J_2$ and $b \in T(X, \rho, R)$. Then $b \in J_2$ if and only if $a \mathcal{J} b$.*

Proof: Suppose $b \in J_2$. Then $\nabla a = \{r, s\}$ and $\nabla b = \{r', s'\}$ for some $r, s, r', s' \in R$. Let $A(r) = \{z \in X : za = r\}$ and $A(s) = \{z \in X : za = s\}$. Since $b \in T(X, \rho, R)$, there are $r'_0, s'_0 \in R$ such that $r'_0 b = r'$ and $s'_0 b = s'$. Define $c \in T(X)$ so that $A(r)c = \{r'_0\}$ and $A(s)c = \{s'_0\}$. By the definition of $A(r)$ and $A(s)$ and the construction of c, we have $c \in T(X, \rho, R)$. Take any $d \in T(X, \rho, R)$ so that $r'd = r$ and $s'd = s$. Then it is clear that $a = cbd$. By symmetry, $b = c'ad'$ for some $c', d' \in T(X, \rho, R)$, and so $a \mathcal{J} b$.

Conversely, suppose that $a \mathcal{J} b$. Then $a = cbd$ for some $c, d \in T(X, \rho, R)$, which implies that $|\nabla a| = |\nabla(cbd)| \leq |\nabla b|$. By symmetry, $|\nabla b| \leq |\nabla a|$, and so $|\nabla b| = |\nabla a| = 2$. Moreover, since $\nabla a \subseteq R$ and $b = c'ad'$ for some $c', d' \in T(X, \rho, R)$, $\nabla b = \nabla(c'ad') \subseteq R$. It follows that $b \in J_2$. ■

Lemma 4.5 *Let A and B be $\mathcal{J}$-classes of $T(X,\rho,R)$ and let $\phi \in Aut(T(X,\rho,R))$. If $\phi(A) \cap B \neq \emptyset$ then $\phi(A) = B$.*

Proof: Since $\phi \in Aut(T(X,\rho,R))$, it follows immediately from the definition of the relation $\mathcal{J}$ that, for all $a,b \in T(X,\rho,R)$, we have

$$a \,\mathcal{J}\, b \Leftrightarrow \phi(a) \,\mathcal{J}\, \phi(b).$$

Suppose $\phi(A) \cap B \neq \emptyset$, that is, there is $a \in A$ such that $\phi(a) \in B$. Let $b \in A$. Then $\phi(a) \,\mathcal{J}\, \phi(b)$ (since $a \,\mathcal{J}\, b$) and so $\phi(b) \in B$ (since B is a $\mathcal{J}$-class and $\phi(a) \in B$). It follows that $\phi(A) \subseteq B$. For the reverse inclusion, observe that $\phi(A) \cap B \neq \emptyset$ implies $A \cap \phi^{-1}(B) \neq \emptyset$. Thus, by the foregoing argument, $\phi^{-1}(B) \subseteq A$, and so $B \subseteq \phi(A)$. ■

Lemma 4.6 *Let $\phi \in Aut(T(X,\rho,R))$. Then $\phi(J_2) \cap J_1 = \emptyset$ and $\phi(J_1) \cap J_2 = \emptyset$.*

Proof: If $J_1 = \emptyset$ or $J_2 = \emptyset$ then the result is clearly true. Thus we may assume that $J_1 \neq \emptyset$ and $J_2 \neq \emptyset$. Suppose, by way of contradiction, that $\phi(J_2) \cap J_1 \neq \emptyset$, that is, there is $a \in J_2$ such that $\phi(a) \in J_1$. Then, since J_1 and J_2 are $\mathcal{J}$-classes of $T(X,\rho,R)$, it follows by Lemma 4.5 that $\phi(J_2) = J_1$. Select $r,s \in R$ such that $r \neq s$ (such r and s exist since $J_2 \neq \emptyset$) and consider $b \in T(X,\rho,R)$ such that

$$b = \begin{pmatrix} r\rho & X - r\rho \\ s & r \end{pmatrix}.$$

Note that $b \in J_2$, b is not an idempotent, and $b^2 \in J_2$. Let $c = \phi(b)$. Since $\phi(J_2) = J_1$, we have $c \in J_1$, c is not an idempotent, and $c^2 \in J_1$. Since $c \in J_1$,

$$c = \begin{pmatrix} A_1 & A_2 \\ t & x \end{pmatrix},$$

where $A_1, A_2 \subseteq X$, $t \in R$, and $x \in t\rho - R$. Since c is not an idempotent, $x \notin A_2$ (note that $t \in A_1$ since $t \in R$ and $Rc \subseteq R$). Thus $x \in A_1$ and so $yc^2 = t$ for every $y \in X$. This is a contradiction since $c^2 \in J_1$ and so $|\nabla c^2| = 2$. Hence $\phi(J_2) \cap J_1 = \emptyset$.

By the foregoing argument, $\phi^{-1}(J_2) \cap J_1 = \emptyset$, which implies $J_2 \cap \phi(J_1) = \emptyset$. ■

Lemma 4.7 *Let $a \in J_1$, $b \in J_2$, and $c \in T(X,\rho,R)$. Then $ac \notin J_2$ and $bc \notin J_1$.*

Proof: Since $a \in J_1$, $\nabla a = \{r,x\}$ where $r \in R$ and $(r,x) \in \rho$. Thus $\nabla(ac) = \{rc, xc\}$ and, since $c \in T(X,\rho,R)$, $(rc,xc) \in \rho$. It follows that $|\nabla(ac) \cap R| = 1$ and so $ac \notin J_2$.

Since $b \in J_2$, $\nabla b \subseteq R$. Thus, since $c \in T(X,\rho,R)$, $\nabla(bc) \subseteq R$ and so $bc \notin J_1$. ■

Theorem 4.8 *The sets J_1, J_2, and T_2 are characteristic in $T(X,\rho,R)$.*

Proof: Since the empty set is characteristic in any semigroup, we may assume that $J_1 \neq \emptyset$ and $J_2 \neq \emptyset$. Let $\phi \in Aut(T(X,\rho,R))$. Let $b \in J_1$. Since ϕ is onto, $b = \phi(a)$ for some $a \in T(X,\rho,R)$. Note that $|\nabla a| \geq 2$ (since $\phi(T_1) = T_1$ by Theorem 4.2). Thus we can find $c \in T(X,\rho,R)$ such that $ac \in T_2$. We have:

$$|\nabla \phi(ac)| = |\nabla(\phi(a)\phi(c))| \leq |\nabla \phi(a)| = |\nabla b| = 2.$$

Thus $\phi(ac) \in T_2$ (since $\phi(T_1) = T_1$). Hence, as $\phi(a) = b \in J_1$ and $\phi(ac) = \phi(a)\phi(c) \notin J_2$ (by Lemma 4.7), we have $\phi(ac) \in J_1$. Therefore, since $\phi(J_2) \cap J_1 = \emptyset$ (by Lemma 4.6), $ac \notin J_2$. It follows that $ac \in J_1$ and $\phi(J_1) \cap J_1 \neq \emptyset$. Hence, since J_1 is a $\mathcal{J}$-class, Lemma 4.5 implies $\phi(J_1) = J_1$.

Using the identical argument (with the roles of J_1 and J_2 interchanged), we can prove that $\phi(J_2) = J_2$. Since $T_2 = J_1 \cup J_2$, it follows that $\phi(T_2) = T_2$. ■

5 Right Inverse Diagonal Bands

A band is a semigroup of idempotents. A *diagonal band* is a band B with zero, 0, such that for all $a, b \in B$:

$$a \neq b \Rightarrow ab = 0.$$

Let $a, b \in J_1$. We say that $(a, b) \in \mathcal{R}^{J_1}$ if and only if there exist $c, d \in J_1$ such that $a = bc$ and $b = ad$.

Lemma 5.1 *Let $e, f \in J_1$ be idempotents. Then $(e, f) \in \mathcal{R}^{J_1}$ if and only if* $\mathrm{Ker}(e) = \mathrm{Ker}(f)$.

Proof: Suppose $(e, f) \in \mathcal{R}^{J_1}$. Then $e = fc$ for some $c \in J_1$ and so $\mathrm{Ker}(e) = \mathrm{Ker}(fc) \subseteq \mathrm{Ker}(f)$. By symmetry, $\mathrm{Ker}(f) \subseteq \mathrm{Ker}(e)$, and so $\mathrm{Ker}(e) = \mathrm{Ker}(f)$.

Conversely, suppose $\mathrm{Ker}(e) = \mathrm{Ker}(f)$. Since e and f are idempotents in J_1, $\nabla e = \{x, r\}$ and $\nabla f = \{y, s\}$, where $x, y \in X - R$, $r, s \in R$, $xe = x$, and $yf = y$. Then the partition induced by $\mathrm{Ker}(e) = \mathrm{Ker}(f)$ must have the form $\{\{x, y, \ldots\}, \{r, s, \ldots\}\}$. (Indeed, suppose $ye = r$. Then $(r, y) \in \mathrm{Ker}(e) = \mathrm{Ker}(f)$ and so $y = yf = rf \in R$, which is a contradiction.) Thus we have

$$e = \begin{pmatrix} \{x, y, \ldots\} & \{r, s, \ldots\} \\ x & r \end{pmatrix} \text{ and } f = \begin{pmatrix} \{x, y, \ldots\} & \{r, s, \ldots\} \\ y & s \end{pmatrix}.$$

Consider the following element of $T(X, \rho, R)$:

$$c = \begin{pmatrix} \{x\} & X - \{x\} \\ y & s \end{pmatrix}.$$

It is obvious that $c \in J_1$ (since $(y, s) \in \rho$) and that $f = ec$. By symmetry, $e = fd$ for some $d \in J_1$, and so $(e, f) \in \mathcal{R}^{J_1}$. ■

Lemma 5.1 implies that $\mathcal{R}^{J_1}$ restricted to $E(J_1) \times E(J_1)$, where $E(J_1)$ is the set of idempotents of J_1, is an equivalence relation on $E(J_1)$, and so it induces a partition of $E(J_1)$. Thus for $e \in E(J_1)$, we can speak of the $\mathcal{R}^{J_1}$-class of e in J_1 (consisting of all $f \in E(J_1)$ such that $(e, f) \in \mathcal{R}^{J_1}$).

Let B be a diagonal band and let 0 be the zero in B. We say that B is a *diagonal band in* J_1 if $B - \{0\} \subseteq J_1$ and $0 \in T_1$. A diagonal band B in J_1 is said to be to be *right inverse* if for every $0 \neq e \in B$, the $\mathcal{R}^{J_1}$-class of e in J_1 has only one idempotent (e itself).

For $x \in X - R$ and $r \in R$ such that $(x, r) \in \rho$, we denote by $t(x, r)$ the element of $T(X, \rho, R)$ defined by: $xt(x, r) = x$ and $(X - \{x\})t(x, r) = \{r\}$. For $r \in R$, we denote by

t_r the element of $T(X,\rho,R)$ whose image is $\{r\}$. It is clear that $t(x,r)$ is an idempotent in J_1 and that $t_r \in T_1$. Finally, for $r \in R$, we define the set $T(r)$ by

$$T(r) = \{t(x,r) : x \in r\rho\} \cup \{t_r\}.$$

Note that if $r\rho = \{r\}$ then $T(r) = \{t_r\}$.

Lemma 5.2 *For every $r \in R$, $T(r)$ is a maximal right inverse diagonal band in J_1.*

Proof: Let $t(x,r), t(y,r) \in T(r)$. Then $t(x,r)t(x,r) = t(x,r)$ and if $x \neq y$ then $t(x,r)t(y,r) = t_r$. Thus $T(r)$ is a diagonal band with zero t_r, and so, since every $t(x,r) \in J_1$, $T(r)$ is a diagonal band in J_1.

We prove that $T(r)$ is right inverse. Note that $\mathrm{Ker}(t(x,r)) = \{\{x\}, X-\{x\}\}$. Suppose $e \in J_1$ is an idempotent such that $(e, t(x,r)) \in \mathcal{R}^{J_1}$. Then, by Lemma 5.1, $\mathrm{Ker}(e) = \mathrm{Ker}(t(x,r))$, and so

$$e = \begin{pmatrix} \{x\} & X-\{x\} \\ y & s \end{pmatrix}$$

for some $s \in R$ and $y \in X - R$. Since e is an idempotent, $ye = y$ and so $y = x$. Moreover, since $y \in s\rho$ and $y = x \in r\rho$, we have $r\rho = s\rho$ and so $r = s$. Thus $e = t(x,r)$.

It remains to prove that $T(r)$ is maximal. Let B be a right inverse diagonal band in J_1 such that $T(r) \subseteq B$. Then $t_r \in B$ and so t_r is the zero in B. Let $t_r \neq e \in B$. Then $e \in J_1$ and so

$$e = \begin{pmatrix} A_1 & A_2 \\ x & s \end{pmatrix},$$

where $x \in X - R$ and $s \in R$. Since e is an idempotent, $x \in A_1$ and $s \in A_2$. Suppose there is $y \in A_1$ such that $y \neq x$. Observe that $A_1 \cap R = \emptyset$ (since $Re \subseteq R$ and $x \notin R$). Thus $r_y \in A_2$ and so

$$f = \begin{pmatrix} A_1 & A_2 \\ y & r_y \end{pmatrix}$$

is an idempotent in J_1 such that $\mathrm{Ker}(e) = \mathrm{Ker}(f)$. This is a contradiction since $f \neq e$ and B is right inverse. It follows that $A_1 = \{x\}$ and so $e = t(x,s)$. Since t_r is the zero in B, $t_r t(x,s) = t_r$. Thus $s = r$ and so $e = t(x,r) \in T(r)$. It follows that $T(r) = B$. ■

Lemma 5.3 *Let B be a maximal right inverse diagonal band in J_1. Then there is $r \in R$ such that $B = T(r)$.*

Proof: Let $r \in R$ be such that t_r is the zero in B. Let $t_r \neq e \in B$ with $\nabla e = \{x, s\}$ ($x \in X - R$, $s \in R$). By the last part of the proof of Lemma 5.2 (the part that shows that $T(r)$ is maximal), $e = t(x,r)$. It follows that $B \subseteq T(r)$, and so, since B is maximal, $B = T(r)$. ■

By Lemma 4.7 and the fact that $|\nabla(ab)| \leq |\nabla a|$ for all $a, b \in T(X)$, $J_1 \cup T_1$ is a subsemigroup of $T(X,\rho,R)$. Thus we can talk about the automorphism group of $J_1 \cup T_1$.

Lemma 5.4 *Let $\phi \in Aut(J_1 \cup T_1)$. Then for every $r \in R$ there is $s \in R$ such that $\phi(T(r)) = T(s)$.*

Proof: Since $\phi \in Aut(J_1 \cup T_1)$ and $T(r)$ is a diagonal band, $\phi(T(r))$ is a diagonal band. Since t_r is the zero in $T(r)$, $\phi(t_r)$ is the zero in $\phi(T(r))$. By Theorem 4.2, $\phi(t_r) \in T_1$ and so $\phi(t_r) = t_s$ for some $s \in R$. Suppose $t_p \in \phi(T(r))$ where $p \in R$. Then $t_s t_p = t_s$ (since t_s is the zero in $\phi(T(r))$). Thus $p = p(t_s t_p) = pt_s = s$ and so $t_p = t_s$. It follows that $\phi(T(r)) - \{t_s\} \subseteq J_1$ and so $\phi(T(r))$ is a diagonal band in J_1.

We prove that $\phi(T(r))$ is right inverse. Since $\phi \in Aut(J_1 \cup T_1)$, it follows from the definition of $\mathcal{R}^{J_1}$ and the fact that T_1 is characteristic in $J_1 \cup T_1$ that for all $a, b \in J_1$, $(a, b) \in \mathcal{R}^{J_1}$ if and only if $(\phi(a), \phi(b)) \in \mathcal{R}^{J_1}$. Moreover, for every $e \in J_1 \cup T_1$, e is an idempotent in J_1 if and only if $\phi(e)$ is an idempotent in J_1. It follows that for every $0 \neq f \in \phi(T(r))$, the $\mathcal{R}^{J_1}$-class of f has only one idempotent.

We proved that $\phi(T(r))$ is a right inverse diagonal band in J_1. Since t_s is the zero in $\phi(T(r))$, it follows by the proof of Lemma 5.3 that $\phi(T(r)) \subseteq T(s)$. Hence $T(r) \subseteq \phi^{-1}(T(s))$. By the foregoing argument, $\phi^{-1}(T(s))$ is a right inverse diagonal band in J_1. Since $T(r)$ is a maximal right inverse diagonal band in J_1 (by Lemma 5.2), it follows that $T(r) = \phi^{-1}(T(s))$, and so $\phi(T(r)) = T(s)$. ■

6 Automorphism Group of $T(X, \rho, R)$

Let $\phi \in Aut(J_1 \cup T_1)$. By Lemma 5.4, for every $x \in X - R$ there is $y \in X - R$ such that $\phi(t(x, r_x)) = t(y, r_y))$, and for every $r \in R$ there is $s \in R$ such that $\phi(t_r) = t_s$.

Thus we can define $g_\phi \in T(X)$ by: $xg_\phi = y$ if $\phi(t(x, r_x)) = t(y, r_y)$ $(x \in X - R)$ and $rg_\phi = s$ if $\phi(t_r) = t_s$ $(r \in R)$.

Note that if $\phi(t(x, r_x)) = t(y, r_y)$ then $\phi(t_{r_x}) = t_{r_y}$. Thus $(xg_\phi, r_x g_\phi) = (y, r_y) \in \rho$.

Lemma 6.1 *Let $\phi \in Aut(J_1 \cup T_1)$. Then $g_\phi \in S(X, \rho, R)$.*

Proof: It is an easy exercise to show that $g_\phi \in S(X)$. We prove that $g_\phi \in T(X, \rho, R)$. Let $r \in R$. Then, since $\phi(T_1) = T_1$, $\phi(t_r) = t_s$ for some $s \in R$. Thus $rg_\phi = s \in R$ and so $Rg_\phi \subseteq R$.

Let $(x, y) \in \rho$. We want to prove that $(xg_\phi, yg_\phi) \in \rho$. If $x, y \in R$ then $x = y$, and so $(xg_\phi, yg_\phi) = (xg_\phi, xg_\phi) \in \rho$. If $x \in X - R$ and $y \in R$ then $y = r_x$, and so $(xg_\phi, yg_\phi) = (xg_\phi, r_x g_\phi) \in \rho$ (by the observation before the lemma). Thus we may assume that $x, y \in X - R$. Since $(x, y) \in \rho$, $r_x = r_y$. Again by the observation before the lemma, $(xg_\phi, r_x g_\phi) \in \rho$ and $(yg_\phi, r_x g_\phi) = (yg_\phi, r_y g_\phi) \in \rho$. Thus, since ρ is symmetric and transitive, $(xg_\phi, yg_\phi) \in \rho$. Hence $g_\phi \in T(X, \rho, R)$ and the result follows from Theorem 3.1. ■

For a function $f : A \to B$ and $A' \subseteq A$, $f \mid A'$ denotes the restriction of f to A'.

Lemma 6.2 *Let S be a semigroup such that $J_1 \cup T_1 \leq S \leq T(X, \rho, R)$ and J_1 is characteristic in S, and let $\phi, \psi \in Aut(S)$. Suppose that $\phi \mid T(r) = \psi \mid T(r)$ for every $r \in R$. Then $\phi = \psi$.*

Proof: Let $\tau = \phi\psi^{-1}$. Then $\tau \in Aut(S)$ and $\tau \mid T(r) = id_{T(r)}$ for every $r \in R$. Note that $\phi = \psi$ if and only if $\tau = id_S$. Thus it remains to prove that $\tau(a) = a$ for every $a \in S$.

Let $a \in S$ and $x \in X$. We need to show that $x\tau(a) = xa$. Suppose $x = r \in R$ and let $s = ra$. Then $t_r a = t_s$ and so $t_r \tau(a) = \tau(t_r)\tau(a) = \tau(t_r a) = \tau(t_s) = t_s$. Thus

$$r\tau(a) = (rt_r)\tau(a) = r(t_r\tau(a)) = rt_s = s = ra.$$

Suppose $x \in X - R$ and let $b = t(x, r_x)a$. We claim that $\tau(b) = b$. First observe that $xb = xt(x, r_x)a = xa$ and $zb = zt(x, r_x)a = r_x a = r_{xa}$ for all $z \in X - \{x\}$.

If $xa = r_{xa}$ then $xb = xa = r_{xa}$ and hence $b = t_{xa}$, which implies $\tau(b) = b$.

Suppose $xa \neq r_{xa}$. Then $\nabla b = \{xa, r_{xa}\}$ and so $b \in J_1$. Moreover, $t(x, r_x)b = b$ and $bt(xa, r_{xa}) = b$. Thus, since τ fixes $t(x, r_x)$ and $t(xa, r_{xa})$, we have $t(x, r_x)\tau(b) = \tau(b)$ and $\tau(b)t(xa, r_{xa}) = \tau(b)$.

Suppose $x\tau(b) \neq xa$. Then $x\tau(b) = x(\tau(b)t(xa, r_{xa})) = r_{xa}$ and for every $z \in X - \{x\}$, $z\tau(b) = z(t(x, r_x)\tau(b)) = r_x\tau(b) \in R$. Thus $\nabla\tau(b) \subseteq R$ and so $\tau(b) \notin J_1$. This is a contradiction since $b \in J_1$ and J_1 is characteristic in S.

Thus $x\tau(b) = xa$. For every $z \in X - \{x\}$, $z\tau(b) = z(t(x, r_x)\tau(b)) = r_x\tau(b) = r_x(\tau(b)t(xa, r_{xa})) = r_{xa}$ (since $r\tau(b) \in R$ and $Rt(xa, r_{xa}) = \{r_{xa}\}$). It follows that $\tau(b) = b$. Hence

$$x\tau(a) = x(t(x, r_x)\tau(a)) = x\tau(t(x, r_x)a) = x\tau(b) = xb = xa.$$

Thus $\tau(a) = a$, which concludes the proof. ■

Let $g \in S(X, \rho, R)$. Denote by λ^g the inner automorphism of $T(X, \rho, R)$ induced by g, that is, $\lambda^g(a) = g^{-1}ag$ for every $a \in T(X, \rho, R)$. Let $\tau^g = \lambda^g \mid (J_1 \cup T_1)$. Since $J_1 \cup T_1$ is characteristic in $T(X, \rho, R)$, τ^g is an inner automorphism of $J_1 \cup T_1$ induced by g, that is, $\tau^g(a) = g^{-1}ag$ for every $a \in T_1 \cup J_1$.

The next lemma shows that every automorphism of $T_1 \cup J_1$ is an inner automorphism induced by a unit of $T(X, \rho, R)$.

Lemma 6.3 *$Aut(J_1 \cup T_1) = \{\tau^g : g \in S(X, \rho, R)\}$.*

Proof: Let $\phi \in Aut(T_1 \cup J_1)$. Then, by Lemma 6.1, $g = g_\phi \in S(X, \rho, R)$. We claim that $\phi = \tau^g$. By Theorem 4.2, T_1 is characteristic in $J_1 \cup T_1$. Thus J_1 is characteristic in $J_1 \cup T_1$ and so, by Lemma 6.2, it suffices to show that $\phi \mid T(r) = \tau^g \mid T(r)$ for every $r \in R$.

Let $r \in R$. Then $\phi(t_r) = t_s$ for some $s \in R$. By the definition of g_ϕ, $rg = s$. Thus for every $x \in X$, $x(\tau^g(t_r)) = x(g^{-1}t_r g) = rg = s$, and so $\tau^g(t_r) = t_s$. Hence, by Lemma 5.4, $\phi(T(r)) = T(s) = \tau^g(T(r))$, and so $\phi \mid T(r) = \tau^g \mid T(r)$. This concludes the proof. ■

We can now prove the main theorems of the paper. The first theorem says that the automorphisms of $T(X, \rho, R)$ are the inner automorphisms induced by the units of $T(X, \rho, R)$.

Theorem 6.4 *$Aut(T(X, \rho, R)) = \{\lambda^g : g \in S(X, \rho, R)\}$.*

Proof: Let $\phi \in Aut(T(X, \rho, R))$. Since $J_1 \cup T_1$ is characteristic in $T(X, \rho, R)$, $\phi \mid (J_1 \cup T_1)$ is an automorphism of $J_1 \cup T_1$. Thus, by Lemma 6.3, $\phi \mid (J_1 \cup T_1) = \tau^g$ for some $g \in S(X, \rho, R)$. Hence $\phi \mid (J_1 \cup T_1) = \lambda^g \mid (J_1 \cup T_1)$ and so, since J_1 is characteristic in $T(X, \rho, R)$, $\phi = \lambda^g$ by Lemma 6.2. ■

To prove the second theorem, we shall need a lemma about the center of $T(X, \rho, R)$. For a semigroup S, the *center* $Z(S)$ of S is defined by:

$$Z(S) = \{a \in S : ab = ba \text{ for every } b \in S\}.$$

Lemma 6.5 *Let $a \in Z(T(X,\rho,R))$. Then for every $x \in X$, $xa = x$ or $xa = r_x$.*

Proof: Let $x \in X$ and let $r = r_x$. If $x = r$ then $xa = ra = r(t_r a) = r(at_r) = r = x$. If $x \neq r$ then $xa = x(t(x,r)a) = x(at(x,r)) \in \nabla t(x,r) = \{x,r\}$. The result follows. ■

The final theorem of the paper says that the automorphism group of $T(X,\rho,R)$ is isomorphic to the group of units of $T(X,\rho,R)$. (The group of units of $T(X,\rho,R)$ is described in Theorem 3.1.)

Theorem 6.6 *$Aut(T(X,\rho,R))$ is isomorphic to $S(X,\rho,R)$.*

Proof: Define $f : S(X,\rho,R) \to Aut(T(X,\rho,R))$ by $f(g) = \lambda^{g^{-1}}$. We claim that f is a group isomorphism. Let $g,h \in S(X,\rho,R)$. Then for every $a \in T(X,\rho,R)$,

$$\lambda^{(gh)^{-1}}(a) = (gh)a(gh)^{-1} = g(hah^{-1})g^{-1} = \lambda^{g^{-1}}(\lambda^{h^{-1}}(a)) = (\lambda^{g^{-1}}\lambda^{h^{-1}})(a).$$

Thus $f(gh) = f(g)f(h)$ and so f is a homomorphism. By Theorem 6.4, f is onto. To show that f is one-to-one, let $g \in \text{Ker}(f)$. Then $f(g) = \lambda^{g^{-1}}$ is the identity automorphism of $T(X,\rho,R)$. Let $a \in T(X,\rho,R)$. Then $\lambda^{g^{-1}}(a) = a$, and so we have:

$$\lambda^{g^{-1}}(a) = a \Rightarrow gag^{-1} = a \Rightarrow ga = ag.$$

It follows that $g \in Z(T(X,\rho,R))$. Thus, since g is a bijection, Lemma 6.5 implies that g is the identity transformation on X. Hence the kernel of f is trivial and so f is one-to-one. ■

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